Nontrivially realized simple symmetries of a simple system
Proste symetrie prostego układu nietrywialnie realizowane

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Abstract. Using the example of a particle moving in infinite space under the influence of a constant in space but possibly time dependent force we discuss the application of the Noether theorem and its implications for the classical and quantum versions of the theory of such a system.
Keywords: Symmetries, Noether theorem, classical mechanics, quantum mechanics, central charges

Abstrakt. Na przykładzie cząstki poruszającej sie w nieskończonej przestrzeni pod wpływem siły niezależnej od położenia, ale niekoniecznie niezależnej od czasu, omawiam zastosowanie twierdzenia Noether i jego konsekwencje dla klasycznej i kwantowej teorii układu fizycznego.
Slowa kluczowe: symetrie, twierdzenie Noether, mechanika klasyczna, mechanika kwantowa, ładunki centralne

1. Introduction

It is often glibly repeated that homogeneity of space understood as symmetry of a given mechanical system with respect to its translations in space, implies conservation of the linear momentum (of the considered mechanical system). While in most cases this is indeed so, we want to point out here that this is not universally true and the conserved quantity associated with a symmetry is unambiguously given by the Noether theorem. Similarly it is usually taken for granted that in quantum mechanics operators realizing a symmetry of a given physical system commute with the system's Hamiltonian and have, therefore, direct bearing on the energy spectrum. This again is not always so and it is the purpose of this note to discuss these issues on an example of a very simple physical system. The system is a particle of mass m moving under the influence of a constant in space, but possibly time-dependent force. The symmetries which will be considered are translations in space and (Galilean) boosts. The example is thus extremely simple and can easily be discussed in classes on classical and quantum mechanics.

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2. Classical Mechanics and Noether theorem

In classical mechanics one speaks of a symmetry of a given physical system if, after it is subjected to an operation S (active view), it "works" the same way as before the operation. The mathematical expression of this operational definition is that the variables parametrizing the state of the original system and of the transformed one satisfy the same equations of motion. For instance, if $\mathbf{r}(t)$ is a trajectory of a pointlike mass m in the gravitational field $-GMm\mathbf{r}/|\mathbf{r}|^3$ of a mass M fixed at the origin of the space, i.e. a trajectory satisfying the equation $m\ddot{\mathbf{r}}(t) = -GMm\mathbf{r}(t)/|\mathbf{r}(t)|^3$, then the trajectory $\mathbf{r}'(t)$ which is obtained by rigidly rotating in space the original trajectory $\mathbf{r}(t)$ by any angle around any axis passing through the center of the force will also satisfy the same Newton's equation: $m\ddot{\mathbf{r}}'(t) = -GMm\mathbf{r}'(t)/|\mathbf{r}'(t)|^3$. Rotations are therefore (continuous) symmetries of this system and one knows that the associated conserved quantity is the angular momentum $\mathbf{L} = m\mathbf{r}(t) \times \dot{\mathbf{r}}(t)$.

In general, conserved quantities associated with continuous symmetries of physical systems, the equations of motion of which follow from the Hamilton's stationary action principle (see e.g. [1]), are unambiguously given by the Noether theorem which says that to each one-parameter family of continuous symmetries of the system corresponds one constant of motion. In the invoked example the Lagrangian of the particle moving in the gravitational field simply does not change under (infinitesimal) rotations $\mathbf{r}(t) \to \mathbf{r}'(t) = \mathbf{r}(t) + \boldsymbol{\theta} \times \mathbf{r}(t)$ of the trajectory: $L(\mathbf{r}', \dot{\mathbf{r}}') = L(\mathbf{r}, \dot{\mathbf{r}})$. This means that also the value I of the action obtained for a given trajectory and the action value I' obtained for the rotated trajectory are equal. Consequently, if I is stationary for some trajectory it must be also stationary for its rotated counterpart. The conserved quantity follows then immediately by considering the equality $I' - I = 0$ expanded to the first order in the parameters $\boldsymbol{\theta}$ of the rotation. The equality $I' = I$ is, however, too strong a condition. I' is stationary whenever I is also if the Lagrangian transforms into a total time derivative of a function depending only on the trajectory $\mathbf{r}(t)$. In such cases $I' = I + \Delta I$ but because one admits only trajectories with fixed ends, the term ΔI does invalidate the conclusion that if I is stationary for some trajectory, I' is also such for the transformed one and again the conserved quantity is identified by considering the equality $I' - I - \Delta I = 0$ expanded to the first order in the parameters of the transformation.

In agreement with the operational understanding of symmetries explained above, if a particle moves under the influence of a (possibly time-dependent but) constant in space force $\mathbf{F}(t)$, translations are symmetries of the system because the translated trajectory satisfies the same equation of motion. It is, however, also clear that the linear momentum $\mathbf{p} = m\dot{\mathbf{r}}(t)$ is not constant, contrary to the standard statement quoted in the introduction. Since translations form a continuous group of symmetries, a conserved quantity should, however, exist. To find it, one has to apply the Noether theorem. The action I corresponding to the considered physical system reads

$$\begin{aligned} I[\mathbf{r}] &= \int_{t_0}^{t_1} dt L(\mathbf{r}, \dot{\mathbf{r}}, t) \\ &= \int_{t_0}^{t_1} dt \left[\frac{1}{2} m\dot{\mathbf{r}}^2 + \mathbf{r}\cdot\mathbf{F}(t)\right]. \end{aligned} \tag{1}$$

Under translations $\mathbf{r}(t) \to \mathbf{r}'(t) = \mathbf{r}(t) + \mathbf{a}$

$$\begin{aligned} L(\mathbf{r}', \dot{\mathbf{r}}', t) &= \frac{1}{2} m\dot{\mathbf{r}}^2 + (\mathbf{r} + \mathbf{a})\cdot\mathbf{F}(t) \\ &= L(\mathbf{r}, \dot{\mathbf{r}}, t) + \frac{d}{dt}\int_{t_0}^{t} dt' \mathbf{a}\cdot\mathbf{F}(t') \\ &\equiv L(\mathbf{r}, \dot{\mathbf{r}}, t) + \frac{d}{dt} X(t), \end{aligned} \tag{2}$$

and, therefore,

$$I' \equiv \int_{t_0}^{t_1} dt L(\mathbf{r}', \dot{\mathbf{r}}', t) = I[\mathbf{r}] + \int_{t_0}^{t_1} dt \frac{d}{dt} X(t). \tag{3}$$

This, as explained above, still ensures that $\mathbf{r}'(t)$ is a solution of the equations of motion if $\mathbf{r}(t)$ is, because if the trajectory $\mathbf{r}(t)$ is a stationary point for I so is $\mathbf{r}'(t)$ for I'.

Following the Noether theorem, the corresponding conservation law is obtained by considering an infinitesimal transformation $\mathbf{r}' = \mathbf{r} + \delta\mathbf{r}$ and writing the above equality expanded to the first order in $\delta\mathbf{r}$ in the form

$$\begin{aligned} 0 &= I' - I - \int_{t_0}^{t_1} dt \frac{d}{dt} \delta X(t) \\ &= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial \mathbf{r}} \cdot \delta\mathbf{r} + \frac{\partial L}{\partial \dot{\mathbf{r}}} \cdot \frac{d}{dt}\delta\mathbf{r} - \frac{d}{dt}\delta X(t)\right] + \dots \\ &= \int_{t_0}^{t_1} dt \left\{\left[\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}}\right] \cdot \delta\mathbf{r} \right. \\ &\qquad \left. + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{r}}} \cdot \delta\mathbf{r} - \delta X(t)\right]\right\} + \dots \end{aligned} \tag{4}$$

Thus, with $\delta\mathbf{r} = \delta\mathbf{a}$ and $\delta X(t) = \delta\mathbf{a} \cdot \mathbf{F}(t)$, if $\mathbf{r}(t)$ satisfies the Euler-Lagrange equations, (so that the first square bracket vanishes) the conserved quantity associated with the symmetry with respect to translations of the system is given by the content of the second square bracket. The conserved quantity is, thus, not the momentum $\mathbf{p} \equiv m\dot{\mathbf{r}}$ but

$$\begin{aligned} \mathbf{Q} &= \frac{\partial L}{\partial \dot{\mathbf{r}}} - \int_{t_0}^{t} dt' \mathbf{F}(t') = m\dot{\mathbf{r}}(t) - \int_{t_0}^{t} dt' \mathbf{F}(t') \\ &\equiv \mathbf{p}(t) - \int_{t_0}^{t} dt' \mathbf{F}(t'). \end{aligned} \tag{5}$$

This is an obvious answer which probably can readily be guessed at by any student unbiased by the considerations of Landau & Lifshitz [2] of the homogeneity of space (as a condition for the translational invariance) which according to these authors is equivalent to the independence of the Lagrangian L on $\mathbf{r}$. (That the obtained conserved quantity is not very useful in the classical theory is another story.)

Whether the constant in space force $\mathbf{F}$ varies with time or not, (Galilean) boosts are also symmetries of the considered system. Indeed, the boosted form $\mathbf{r}'(t) = \mathbf{r}(t) + \mathbf{V}t$ of the solution

$$\mathbf{r}(t) = \mathbf{r}(t_0) + v(t_0)(t - t_0) + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \frac{\mathbf{F}(t'')}{m}, \tag{6}$$

of the Newtonian equation of motion $m\ddot{\mathbf{r}} = \mathbf{F}(t)$ is also the solution of this equation. Although the concomitant conserved quantity is rarely mentioned in textbooks on mechanics (in [2] the symmetry with respect to the boosts is only used to constrain the possible form of the Lagrangian of a free particle) it too can be found by applying the Noether theorem. To this end one considers again the action (1) and the change of the Lagrangian

$L(\mathbf{r}, \dot{\mathbf{r}}, t)$ under a boost

$$L(\mathbf{r}', \dot{\mathbf{r}}', t) = L(\mathbf{r} + \mathbf{V}t, \dot{\mathbf{r}}' + \mathbf{V}, t)$$
$$= \frac{1}{2}m\dot{\mathbf{r}}^2 + m\dot{\mathbf{r}}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2 + (\mathbf{r} + \mathbf{V}t)\cdot\mathbf{F}(t)$$
$$= L(\mathbf{r}, \dot{\mathbf{r}}, t) + \frac{d}{dt}\left[m\mathbf{r}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2 t + \int_{t_0}^{t} dt' t' \mathbf{F}(t')\cdot\mathbf{V}\right]. \quad (7)$$

It is crucial that the quantity X, defined as the content of the square bracket, depends only on $\mathbf{r}(t)$ and t (and not on $\dot{\mathbf{r}}(t)$) owing to this, if $\mathbf{r}(t)$ is a stationary point for I so is $\mathbf{r}'(t)$ for I' because the variations $\delta\mathbf{r}(t)$ one admits vanish at $t = t_0$ and $t = t_1$ (the variations of the velocity are not bound).

As follows from the Noether theorem, conserved is the quantity

$$\frac{\partial L}{\partial \dot{\mathbf{r}}}\cdot\delta\mathbf{r} - \delta X = m\dot{\mathbf{r}}\cdot\delta\mathbf{V}t - m\mathbf{r}\cdot\delta\mathbf{V} - \int_{t_0}^{t} dt' t' \, \mathbf{F}(t')\cdot\delta\mathbf{V}\,. \quad (8)$$

Because $\delta\mathbf{V}$ is arbitrary, the conserved vector is

$$\mathbf{K} = t\mathbf{p}(t) - m\mathbf{r}(t) - \int_{t_0}^{t} dt' t' \mathbf{F}(t'). \quad (9)$$

The constancy of $\mathbf{K}$ can be checked directly using the explicit solution (6) of the equation of motion and the obvious solution for $\mathbf{p}(t)$:

$$\mathbf{K} = t\left(\mathbf{p}(t_0) + \int_{t_0}^{t} dt' \mathbf{F}(t')\right) - m\left(\mathbf{r}(t_0) + \frac{\mathbf{p}(t_0)}{m}(t - t_0) + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \frac{\mathbf{F}(t'')}{m}\right) - \int_{t_0}^{t} dt' t' \mathbf{F}(t'). \quad (10)$$

To see that $\mathbf{K}$ is indeed constant, it is sufficient to integrate by parts

$$\int_{t_0}^{t} dt' t' \mathbf{F}(t') = \int_{t_0}^{t} dt' t' \frac{d}{dt'} \int_{t_0}^{t'} dt'' \mathbf{F}(t'') = \left[t' \int_{t_0}^{t'} dt'' \mathbf{F}(t'')\right]_{t_0}^{t} - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \mathbf{F}(t''). \quad (11)$$

Thus, $\mathbf{K}(t) = t_0\mathbf{p}(t_0) - m\mathbf{r}(t_0)$.

Both conserved quantities, $\mathbf{Q}$ and $\mathbf{K}$, will play the role of the symmetry generators in the quantum version of the theory of the considered system.[1]

1. Of course, they play a similar role also classically, generating symmetry transformations through the appropriate Poisson brackets.

3. Quantum Mechanics

Given that symmetries are so important in modern physics (for instance relativistic quantum field theory can be formulated [9] by imposing the requirements of Poincaré symmetry on the theory of interacting particles formulated in the second quantization formalism) it is somewhat surprising to realize that the proper general definition of a symmetry is usually not found in standard quantum mechanics textbooks like [3, 5] and even in books entirely devoted to symmetries like [6]. Even if giving a definition is attempted, it is usually restricted to symmetry operations represented in the Hilbert space by operators which commute with the system's Hamiltonian (see e.g. [7, 8]). Weinberg [9] who devoted an otherwise very interesting and enlightening section to symmetries gives a rather misleading definition (adopting implicitly the passive view) by saying that "A symmetry transformation is a change in our point of view that does not change the result of possible experiments. If an observer $\mathcal{O}$ sees a system in a state represented by a ray $\mathcal{R}$ or $\mathcal{R}_1$ or $\mathcal{R}_2, \ldots$, then an equivalent observer $\mathcal{O}'$ who looks at the *same* system will observe it in a different state represented by a ray $\mathcal{R}'$ or $\mathcal{R}'_1$ or $\mathcal{R}'_2, \ldots$, respectively but the two observers must find the same probabilities: $P(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$." and quotes the Wigner result that such transformations are in the Hilbert space represented by unitary (or antiunitary) operators U satisfying the requirement[2] $|(U\Psi|U\Psi_n)| = |(\Psi|\Psi_n)|$ (Ψ, Ψ_n, Ψ', Ψ'_n are Hilbert space vectors belonging to the rays $\mathcal{R}$, $\mathcal{R}_n$ or $\mathcal{R}'$ and $\mathcal{R}'_n$, respectively). That at least some element is missing in this definition is evident if one takes the unitary operator $U = \exp(-i\mathbf{a}\cdot\hat{\mathbf{P}})$ ($\hat{\mathbf{P}}$ is the momentum operator and $\mathbf{a}$ an arbitrary vector) which according to this definition would represent a symmetry of, say, the system formed by an electron moving in the Coulomb field of a static positive charge.

In fact the missing element is found only in the Schiff textbook [10] from which one can learn that the definition of a symmetry in quantum mechanics is essentially the same as in the classical theory: a symmetry operation S is any such operation that after performing it on the system its "working" remains unchanged. In the language of the quantum mechanics it means that if the state vector $|\Psi(t)\rangle$ of the original system satisfies the Schrödinger equation

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle \quad (12)$$

the state vector $U_S(t)|\Psi(t)\rangle$ representing the system on

2. We use capital characters Ψ instead of ψ to stress that these considerations are general and apply also to relativistic quantum field theories.

which the operation S has been performed satisfies the Schrödinger equation

$$i\hbar\frac{d}{dt}\left(U_S(t)|\Psi(t)\rangle\right) = \hat{H}(t)\,U_S(t)|\Psi(t)\rangle, \qquad (13)$$

with the same Hamiltonian $\hat{H}$. Compared to the definition given in [10] we have explicitly indicated a possible time dependence of the symmetry operator U_S because even if the symmetry operation itself is the same, no matter at which instant it is performed, the operator representing it may depend on time as is the case of the symmetries considered here. From the definition given here it follows that if $U(t_2, t_1)$ is the (Schrödinger picture) evolution operator[3] associated with the Hamiltonian $\hat{H}$ of the system, such that $|\Psi(t_2)\rangle = U(t_2,t_1)|\Psi(t_1)\rangle$, then from (12) and (13) it follows that an operator $U_S(t)$ represents a possible symmetry operation S if

$$U_S(t_2)\,U(t_2,t_1) = U(t_2,t_1)\,U_S(t_1). \qquad (14)$$

In most cases when the Hamiltonian of the system does not depend explicitly on time, $U(t_2,t_1) = \exp(-(i/\hbar)(t_2-t_1)\hat{H})$, and U_S is independent of time, this condition is satisfied because $[U_S, \hat{H}] = 0$. In such cases the symmetry represented by U_S has direct consequences for the Hamiltonian spectrum, but as we want to point out, this is not the most general possibility.

In the case of the simple system considered in this note canonical quantization allows to identify operators ($\hat{1}$ is the unit operator)

$$\hat{\mathbf{Q}} = \hat{\mathbf{p}} - \hat{1}\int_{t_0}^{t} dt'\,\mathbf{F}(t'), \qquad (15)$$

$$\hat{\mathbf{K}} = t\hat{\mathbf{p}} - m\hat{\mathbf{r}} - \hat{1}\int_{t_0}^{t} dt'\,t'\,\mathbf{F}(t'), \qquad (16)$$

which despite being explicitly depend on time are the Schrödinger picture operators (the operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are time independent), as the generators of space translations and boosts. It is convenient already at this point to write down the commutation relations defining the algebra formed by the operators $\hat{\mathbf{Q}}$, $\hat{\mathbf{K}}$ and the Hamiltonian $\hat{H}$

$$\hat{H}(t) = \frac{\hat{\mathbf{p}}^2}{2m} - \hat{\mathbf{r}}\cdot\mathbf{F}(t), \qquad (17)$$

which, when the force is constant, is the generator of the time translation symmetry.[4] Using the standard rules $[\hat{r}^i, \hat{p}^j] = i\hbar\delta^{ij}\hat{1}$, etc. one easily finds the commutators:

$$\begin{aligned}
&[\hat{Q}^i(t), \hat{Q}^j(t')] = 0, \\
&[\hat{K}^i(t), \hat{Q}^j(t')] = -i\hbar m\,\delta^{ij}\hat{1}, \\
&[\hat{H}(t), \hat{Q}^i(t')] = -i\hbar F^i(t)\hat{1}, \\
&[\hat{K}^i(t), \hat{K}^j(t')] = i\hbar m(t-t')\delta^{ij}\hat{1}, \\
&[\hat{H}(t), \hat{K}^i(t')] = i\hbar\hat{p}^i - i\hbar t' F^i(t)\hat{1}.
\end{aligned} \qquad (18)$$

The unitary operators

$$\begin{aligned}
U_Q(\mathbf{a}, t) &= \exp\left(-\frac{i}{\hbar}\mathbf{a}\cdot\hat{\mathbf{Q}}(t)\right), \\
U_K(\mathbf{V}, t) &= \exp\left(-\frac{i}{\hbar}\mathbf{V}\cdot\hat{\mathbf{K}}(t)\right),
\end{aligned} \qquad (19)$$

should realize on the Hilbert space vectors $|\psi\rangle$ representing quantum states of the considered system finite (active) symmetry transformations of space translations and boosts. More precisely, in agreement with the definition of a symmetry formulated above, if $|\psi(t)\rangle$ is an evolving in time as dictates the Schrödinger equation (12) with the Hamiltonian (17) state-vector of the system (of the particle), then $U_Q(\mathbf{a},t)|\psi(t)\rangle$ and $U_K(\mathbf{V},t)|\psi(t)\rangle$ should be the evolving with the time state-vectors of the, respectively, translated in space and boosted systems. That is, $U_Q(\mathbf{a},t)|\psi(t)>$ and $U_K(\mathbf{V},t)|\psi(t)>$ should also be solutions of the same Schrödinger equation. To see that e.g.

$$i\hbar\frac{d}{dt}\left(U_Q(\mathbf{a},t)|\psi(t)\rangle\right) = \hat{H}(t)\left(U_Q(\mathbf{a},t)|\psi(t)\rangle\right), \qquad (20)$$

holds if $|\psi(t)\rangle$ is a solution of the Schrödinger equation, it is sufficient to check that the relation

$$i\hbar\frac{d}{dt}U_Q(\mathbf{a},t) = [\hat{H}(t), U_Q(\mathbf{a},t)], \qquad (21)$$

and the analogous relation with $U_K(\mathbf{V},t)$ do hold. To demonstrate this, we first notice, that the derivative on the left hand side is computed by expanding the operator $U_Q(\mathbf{a},t)$

$$\begin{aligned}
&i\hbar\frac{d}{dt}U_Q(\mathbf{a},t) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\sum_{k=0}^{n-1}(\mathbf{a}\hat{\mathbf{Q}})^k\left[i\hbar\frac{d}{dt}\mathbf{a}\cdot\hat{\mathbf{Q}}(t)\right](\mathbf{a}\hat{\mathbf{Q}})^{n-1-k}.
\end{aligned} \qquad (22)$$

On the other hand, computing the commutator on the right hand side upon using the standard rule $[A, B^n] = [A,B]B^{n-1} + B[A,B]B^{n-2} + \cdots + B^{n-1}[A,B]$ one obtains:

$$\begin{aligned}
&[\hat{H}(t), U_Q(\mathbf{a},t)] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\sum_{k=0}^{n-1}(\mathbf{a}\hat{\mathbf{Q}})^k[\hat{H}(t), \mathbf{a}\hat{\mathbf{Q}}(t)](\mathbf{a}\hat{\mathbf{Q}})^{n-1-k}.
\end{aligned} \qquad (23)$$

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3. Do not confuse symmetry operators $U_S(t)$ with the evolution operator $U(t_2, t_1)$ which has, however, two time arguments.

4. If $\mathbf{F}$ does not depend on time, the classical trajectory $\mathbf{r}'(t) = \mathbf{r}(t-\tau)$ is obviously the solution of the equation of motion $m\ddot{\mathbf{r}} = \mathbf{F}$, provided $\mathbf{r}(t)$ is and, because then the Lagrangian does not depend then on time explicitly, the Hamiltonian is a constant of motion, as is well known from classical mechanics; similarly, $|\psi'(t)\rangle = |\psi(t-\tau)\rangle = e^{i\tau\hat{H}/\hbar}|\psi(t)\rangle$ is obviously the solution of the Schrödinger equation with the time independent Hamiltonian, if $|\psi(t)\rangle$ is.

As the structures of the right hand sides of the formulae (22) and (23) are formally identical, the equalities

$$i\hbar\frac{d}{dt}\left[\mathbf{a}\cdot\hat{\mathbf{Q}}(t)\right] = -i\hbar\,\mathbf{a}\cdot\mathbf{F}(t)\,\hat{1} = [\hat{H}(t), \mathbf{a}\cdot\hat{\mathbf{Q}}(t)],$$
$$i\hbar\frac{d}{dt}\left[\mathbf{V}\cdot\hat{\mathbf{K}}(t)\right] = i\hbar\left(\mathbf{V}\cdot\hat{\mathbf{p}} - t\mathbf{V}\cdot\mathbf{F}(t)\,\hat{1}\right) = [\hat{H}(t), \mathbf{V}\cdot\hat{\mathbf{K}}(t)], \quad (24)$$

which can be directly checked to hold (the commutators on the right hand sides are just the equal-time versions of the ones given in (18)), are sufficient to prove that the requisite relations do indeed hold. Moreover, as the commutator $[\hat{H}(t), \mathbf{a}\cdot\mathbf{Q}(t)]$ is a c-number, one can sum the series to obtain[5]

$$i\hbar\frac{d}{dt}U_Q(\mathbf{a},t) = [\hat{H}(t), U_Q(\mathbf{a},t)] = -\mathbf{a}\mathbf{F}(t)U_Q(\mathbf{a},t). \quad (25)$$

The second equality will be useful in displaying the role of $U_Q(\mathbf{a},t)$ as far as the Hamiltonian spectrum is concerned.

It is also instructive to show that the observables represented by the Hermitian operators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$ are constants of motion even though the operators themselves do not commute with the Hamiltonian. To this end it is most convenient to go over to the Heisenberg picture in which vectors representing states of the system do not change with time and the whole time dependence is carried by the Heisenberg picture $\hat{O}_H(t)$ counterparts of the Schrödinger operators $\hat{O}$. The Heisenberg picture operators are defined by

$$\hat{O}_H(t) = U^\dagger(t,t_0)\,\hat{O}(t)\,U(t,t_0), \quad (26)$$

where it has been assumed that the Schrödinger and Heisenberg pictures coincide at $t = t_0$. To demonstrate the conservation of the quantities (observables) observables represented by the operators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$ it suffices, therefore, to find the Heisenberg picture counterparts of these operators and to check their time independence.

If the force $\mathbf{F}$ is not constant, finding the explicit form of the operator $U(t,t_0)$ is not easy. However this is not necessary. In the considered quantum theory there are two basic operators: the position $\hat{\mathbf{r}}$ and the momentum $\hat{\mathbf{p}}$ operators; all other operators, including $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$, can be built out of these two. (In other words $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ form the basis of the algebra of operators of the considered theory.) It is therefore sufficient to find the Heisenberg form of $\hat{\mathbf{r}}_H(t)$ and $\hat{\mathbf{p}}_H(t)$. This can be done by exploiting the fact that in the considered case the most general form of the classical solution depending on the initial data is known explicitly.[6] Taking t_0 as the moment at which the Schrödinger and the Heisenberg pictures coincide and promoting $\mathbf{r}(t_0)$ and $\mathbf{p}(t_0)$ to Schrödinger picture operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ one obtains[7]

$$\hat{\mathbf{r}}_H(t) = U^\dagger(t,t_0)\,\hat{\mathbf{r}}\,U(t,t_0) = \hat{\mathbf{r}} + \frac{\hat{\mathbf{p}}}{m}(t-t_0) + \hat{1}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\,\frac{\hat{\mathbf{F}}(t'')}{m},$$
$$\hat{\mathbf{p}}_H(t) = U^\dagger(t,t_0)\,\hat{\mathbf{p}}\,U(t,t_0) = \hat{\mathbf{p}} + \hat{1}\int_{t_0}^{t}dt'\,\mathbf{F}(t'). \quad (27)$$

It is straightforward to check that these Heisenberg picture operators satisfy the Heisenberg equations of motion

$$\frac{d}{dt}\hat{O}_H(t) = \frac{i}{\hbar}[\hat{H}^H(t), \hat{O}_H(t)] + U^\dagger(t,t_0)\,\frac{\partial\hat{O}(t)}{\partial t}\,U(t,t_0), \quad (28)$$

in which $\hat{H}^H(t)$ is the Heisenberg picture counterpart of the Hamiltonian and the last term, frequently but somewhat misleadingly denoted by the symbol $(\partial\hat{O}(t)/\partial t)_H$ or even $\partial\hat{O}_H(t)/\partial t$, is just the transformation to the Heisenberg picture of the time derivative of the Schrödinger picture operator. Computing the commutator in the right hand side of the Heisenberg equation (28) using the trick

$$[\hat{H}^H(t), \hat{O}_H(t)] = U^\dagger(t,t_0)\,[\hat{H}(t), \hat{O}(t)]\,U(t,t_0), \quad (29)$$

one easily finds that the right hand sides of the equations satisfied by $\hat{\mathbf{r}}_H(t)$ and $\hat{\mathbf{p}}_H(t)$ read (in these cases the term $(\partial\hat{O}(t)/\partial t)_H = 0$)

$$\frac{\hat{\mathbf{p}}_H(t)}{m} \equiv \frac{\hat{\mathbf{p}}(t)}{m} + \hat{1}\int_{t_0}^{t}dt'\,\frac{\mathbf{F}(t)}{m}, \quad \text{and} \quad \mathbf{F}(t), \quad (30)$$

respectively. This is precisely what is obtained by differentiating with respect to time the explicit forms (27) of $\hat{\mathbf{r}}_H(t)$ and $\hat{\mathbf{p}}_H(t)$.

5. As the analogous commutator of $\hat{H}(t)$ with $\mathbf{a}\cdot\hat{\mathbf{K}}$ is not a c-number, the derivative and of $U_K(\mathbf{V},t)$ and its commutator with $\hat{H}(t)$ cannot be written in a similarly compact form.

6. The possibility of using the known solutions $q^i(t) = q^i(t,q_0,p_0)$, $p_i(t) = p_i(t,q_0,p_0)$ of the classical Hamilton's equations of motion to find directly the form of the Heisenberg picture operators relies implicitly on the fact that the initial data q_0^i and p_{i0} are good canonical variables (the function generating the canonical transformations from $q^i(t)$, $p_i(t)$ to q_0^i and p_{i0} is the appropriately understood action I): if the system is quantized by promoting q_0^i and p_{i0} to the Schrödinger picture operators satisfying the standard commutation relations the relations $q^i(t) = q^i(t,q_0,p_0)$, $p_i(t) = p_i(t,q_0,p_0)$ in which q_0^i and p_{i0} are now operators become just the Heisenberg picture operators.

7. These relations determine $U(t,t_0)$ only up to a (possibly time dependent) phase factor but are sufficient for our purpose.

The Heisenberg picture counterparts of the space-translations and boosts generators (15) and (16) are then given by

$$\begin{aligned}\hat{\mathbf{Q}}_H(t) &= U^\dagger(t,t_0)\,\hat{\mathbf{Q}}(t)\,U(t,t_0)\\ &= \hat{\mathbf{p}}_H(t) - \hat{1}\int_{t_0}^{t} dt'\,\mathbf{F}(t'),\end{aligned} \tag{31}$$

$$\begin{aligned}\hat{\mathbf{K}}_H(t) &= U^\dagger(t,t_0)\,\hat{\mathbf{K}}(t)\,U(t,t_0)\\ &= t\,\hat{\mathbf{p}}_H(t) - m\hat{\mathbf{r}}_H(t) - \hat{1}\int_{t_0}^{t} dt'\,t'\,\mathbf{F}(t').\end{aligned} \tag{32}$$

That $\hat{\mathbf{Q}}_H(t)$ is in fact not dependent on time is readily seen by substituting in it the explicit form of $\hat{\mathbf{p}}_H(t)$. Showing that neither is $\hat{\mathbf{K}}_H(t)$ boils down, after using in the right hand side the explicit forms of $\hat{\mathbf{p}}_H(t)$ and $\hat{\mathbf{r}}_H(t)$, to the same manipulations which showed that classically the quantity $\mathbf{K}$ is constant. Of course the same follows by checking that the right hand sides of the Heisenberg equations of motion (28) of $\hat{\mathbf{Q}}_H(t)$ and $\hat{\mathbf{K}}_H(t)$ vanish (the terms $(\partial\hat{O}(t)/\partial t)_H$ are crucial for this). In more complicated theories, in which obtaining explicit forms of the Heisenberg picture operators is not possible, it is precisely checking (with the help of the trick (29) which makes it possible to use the properties of the Schrödinger picture operators which are always given) that the right hand side of the relevant Heisenberg equation of motion vanishes that tells us that a given (Hermitian) operator represents a quantity that is a constant of motion for a given system. For instance, it is in this way that in models of quantum field theory one checks the constancy of the observables represented by operators generating boosts.

Since the operators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$, despite representing conserved observables, do not commute with the Hamiltonian, the symmetries of the system they generate do not need to have direct implications for the energy spectrum of the system (at a fixed instant of time, if the force $\mathbf{F}$ is not time independent). Nevertheless, in the considered simple situation the spectrum of the system translated in space is related to the original one. Indeed, using the obtained commutator (25) one easily finds that if $\hat{H}(t)|\psi_E(t)\rangle = E(t)|\psi_E(t)\rangle$, then

$$\begin{aligned}&\hat{H}(t)U_Q(\mathbf{a},t)|\psi_E(t)\rangle\\ &\qquad = (E(t) - \mathbf{a}\cdot\mathbf{F}(t))\,U_Q(\mathbf{a},t)|\psi_E(t)\rangle.\end{aligned} \tag{33}$$

In other words, action the space translation operator $U_Q(\mathbf{a},t)$ on a Hamiltonian eigenvector[8] yields another eigenvector but corresponding to another value of the system's energy: the energy of a Hamiltonian (instantaneous) eigenstate translated in space by $\mathbf{a}$ is shifted by $-\mathbf{a}\cdot\mathbf{F}(t)$ compared to the energy of the original state. This is easy to understand: in classical physics it is the force $\mathbf{F}$ which matters and not the potential $V(\mathbf{r})$ which can be changed by adding to it any (constant in space) function of time without affecting the system's motion. In the quantum theory it is the potential that replaces the force, but physical are in fact only differences of energies which are insensitive to changes of the potential by a (time dependent) constant. The system translated by $\mathbf{a}$ should have the same absolute energies as the original one if instead of $V = -\mathbf{r}\cdot\mathbf{F}$ it were placed in the potential $V' = -(\mathbf{r}-\mathbf{a})\cdot\mathbf{F}$. Thus energies of the translated system with respect to the modified Hamiltonian would be the same as energies of the original system with respect to the original Hamiltonian and, hence, are shifted by $-\mathbf{a}\cdot\mathbf{F}$ with respect to the original Hamiltonian.

Finally it is good to notice that the commutation relations of the generators of the symmetries: space translations, boosts and translations in time (if the Hamiltonian is time independent), obtained directly from the rules of composition of symmetry operations assuming that the group of these symmetries is realized in the Hilbert space nonprojectively are different than the ones found here.[9] This only shows that the symmetry group is represented projectively for algebraic reasons:[10] in some of the commutators there appear central charges (terms proportional to the unit operator). If symmetries of a system form the full Galileo group, the central charges in the commutators $[\hat{H},\hat{\mathbf{Q}}]$ and $[\hat{H},\hat{\mathbf{K}}]$ are forbidden by the Jacobi identities involving the generators of the rotations. Rotations, however, are not symmetries in the considered case and it is for this reason that the additional (with respect to the only one in the commutator $[\hat{Q}^i,\hat{K}^j]$ which is allowed by the Jacobi identities involving generators of the full Galileo group and does appear in all known realizations of this group by operators in Hilbert spaces) central charges are possible. To be more precise: the operators $\hat{\mathbf{J}}$ of the angular momentum can, of course, be constructed in the considered case and the Jacobi identities involving them are satisfied (they are algebraic relations which must always hold independently of the question of symmetries), but the forms of the commutators of $\hat{\mathbf{J}}$

8. The considered Hamiltonian has in fact no eigenvectors in the proper Hilbert space (of normalizable states). Its spectrum is continuous and its (instantaneous) eigenvectors corresponding to energies E are generalized vectors (elements of the space dual to the proper Hilbert space) given in the position representation by the appropriate Airy (or Bessel) functions [4].

9. The procedure of finding the commutators of the symmetry generators on the basis of the composition rule of symmetry transformations is given e.g. in [9].

10. Even if the Hilbert space representation of the algebra of the symmetry group generators is free from central charges, the symmetry group itself may still be represented projectively for topological reasons, as happens with the rotation and Poincaré groups.

with the generators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$ are different from those they would have, if they generated good symmetries: the operators $\hat{\mathbf{Q}}$ and $\hat{\mathbf{K}}$, when acted upon by the operators of rotations, do not transform as true vectors (their parts proportional to the unit operator do not rotate) and it is for this reason that the Jacobi identities involving the operators $\hat{\mathbf{J}}$ do not forbid the central charges found in the commutators of the true generators (obtained with the help of the canonical quantization from the Noether theorem) of the symmetries of the system.

Note added. After writing this paper I became aware of the work [11] which discusses the same problem using the same example but formulating the Noether theorem in the framework of the Hamiltonian action principle rather than the Lagrangian one. Its authors do not, however, state explicitly the general condition for the existence of a symmetry of a quantum system and do not relate the algebra of the symmetry generators of the discussed system to the algebra of the Galileo group generators. For this reasons I believe that the present text may still be of some pedagogical value.

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